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MIKE 21

Elliptic Mild-Slope Wave Module

Scientific Documentation

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APPENDICES

1 Introduction

1.1 About this Guide

The purpose of this document is to provide the user the scientific background of the Elliptic Mild-Slope Wave Module, MIKE 21 EMS.

1.2 What Does this Guide Contain?

The entries listed below are included in the present document as separate sections.

- General Description
- Scientific Background
- References

The paper enclosed in Appendix A aims at giving an in-depth description of the physical, mathematical and numerical background related to wave modelling using elliptic mild-slope wave equations and particularly of MIKE 21 EMS.

1.3 General Description

The Elliptic Mild-Slope Wave Module, MIKE 21 EMS, is based on the numerical solution of the so-called 'mild-slope' wave equation originally derived by Berkhoff in 1972. This equation governs the motion of time harmonic water waves of infinitesimal height (linear waves) on a gently sloping bathymetry with arbitrary water depth. In case of constant water depth, the basic equation reduces to the classical Helmholtz equation.

The linear model includes shoaling, refraction, diffraction, wave breaking, bed friction and back-scattering. Partial reflection and transmission through piers and breakwaters is also included. Sponge layers are applied where full absorption of wave energy is required, e.g. at offshore boundaries. Thus, the model is particularly useful to determine wave disturbance in ports and harbours in cases where the forcing wave conditions can be represented by a monochromatic and unidirectional wave. The model has been used to determine harbour resonance and seiching as well as for wave transformation in coastal areas.
The model also includes a general formulation of radiation stresses, which apply in crossing wave trains and in areas of strong diffraction and wave breaking.

MIKE 21 EMS is based on a quite unique solution method. The time-harmonic variation is subtracted and the elliptic equations are reformulated as mass and momentum type equations, which are discretised using a FD scheme. The normal ADI (alternating direction implicit) algorithm is invoked and the equations are solved by means of the double sweep algorithm.

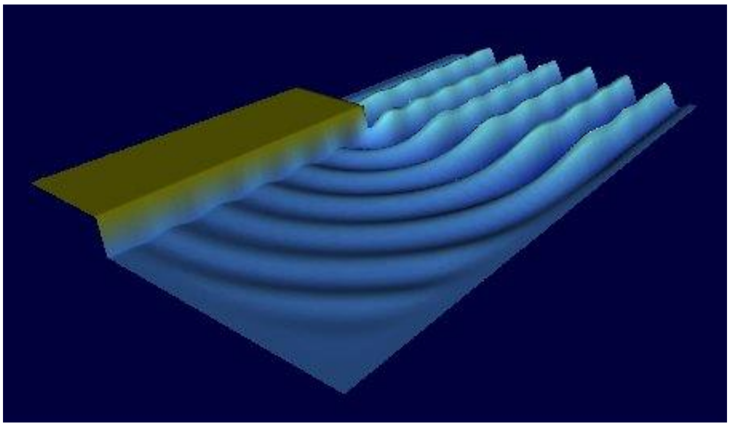

Figure 1.1 Example of MIKE 21 EMS application. The figure shows the instantaneous surface elevation behind a fully reflective breakwater. The incoming waves have a period of 8 s and the water depth is 40 m

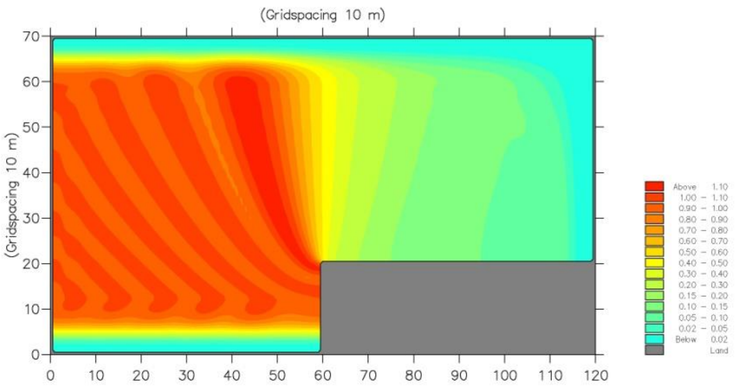

Figure 1.2 Example of MIKE 21 EMS application. The figure shows the wave disturbance coefficient behind a fully reflective breakwater

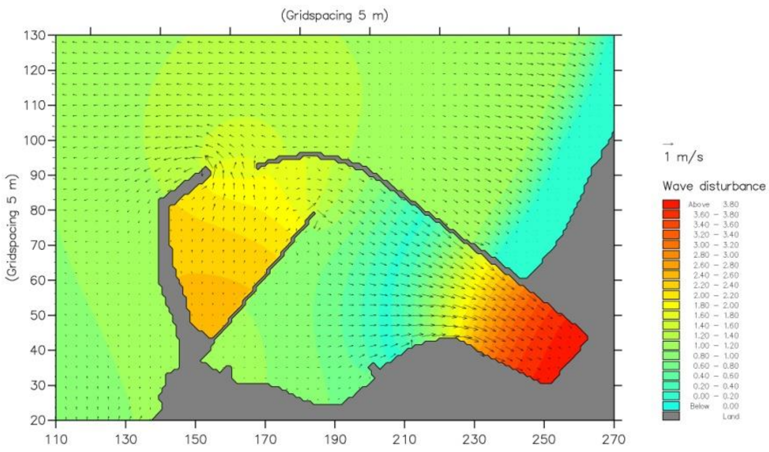

Figure 1.3 Example of MIKE 21 EMS application. The figure shows the wave disturbance coefficient in a harbour basin for a near-resonance condition. Also the instantaneous particle velocities are shown.

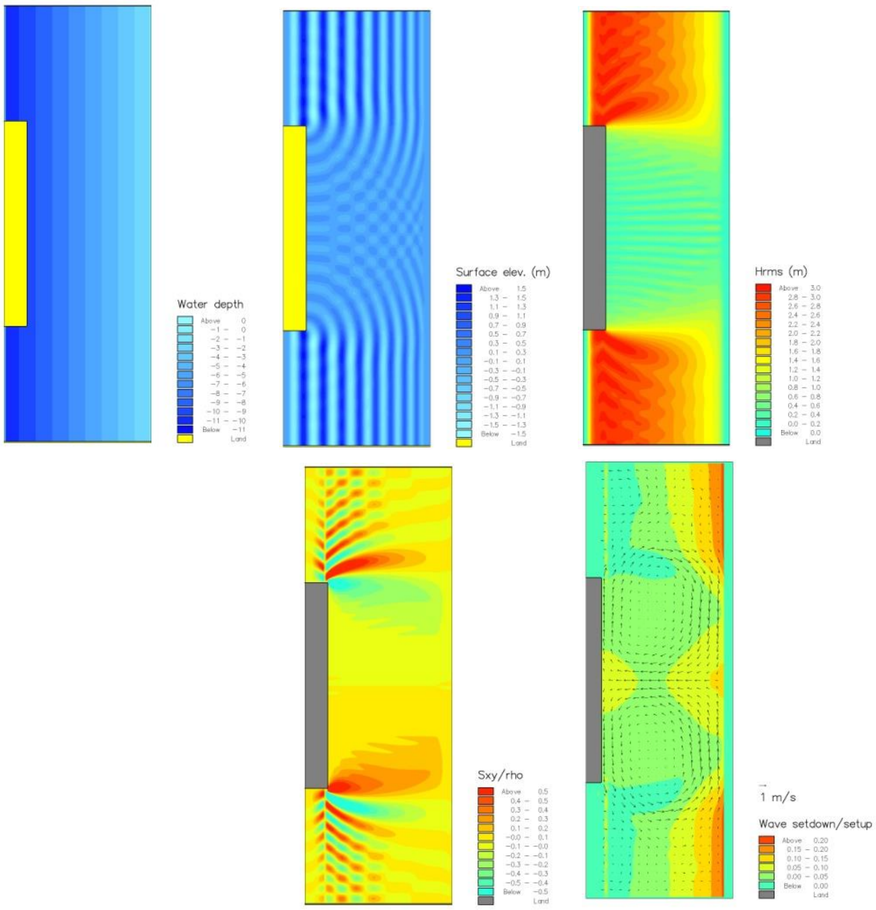

Figure 1.4 Example of MIKE 21 EMS application. The upper left panel shows the bathymetry, the upper middle panel the instantaneous surface elevation and upper right panel the Hrms wave height. The lower left panel shows the radiation stress component Sxy and the lower right panel the wave induced current modelled with MIKE 21 Flow Model. Incident wave conditions: Hrms= 3 m and T= 8 s

2 Scientific Background

The Elliptic Mild-Slope Wave Module of MIKE 21 solves the mild-slope wave equation expressed in two horizontal dimensions. The equation reads:

2.1 Basic Equations

$$\nabla \cdot (cc_g \nabla \zeta) - \frac{c_g}{c}\frac{\partial^2 \zeta}{\partial t^2} = 0 \tag{2.1}$$

where c is the phase celerity, c_g the group velocity[1] and ζ the surface elevation and ∇ the horizontal gradient operator.

By introducing the pseudo fluxes P^* and Q^*, this equation can be rewritten as a system of first order equations, which are similar to the mass and momentum equations governing nearly horizontal flow in shallow water:

$$\frac{c_g}{c}\frac{\partial \zeta}{\partial t} + \frac{\partial P^*}{\partial x} + \frac{\partial Q^*}{\partial y} = 0$$

$$\frac{\partial P^*}{\partial t} + cc_g \frac{\partial \zeta}{\partial x} = 0 \tag{2.2}$$

$$\frac{\partial Q^*}{\partial t} + cc_g \frac{\partial \zeta}{\partial y} = 0$$

The harmonic time variation can now be extracted from the equations by using

$$\zeta = S(x,y,t)e^{i\omega t}$$

$$P^* = P(x,y,t)e^{i\omega t} \tag{2.3}$$

$$Q^* = Q(x,y,t)e^{i\omega t}$$

Now the remaining time variation in S, P and Q is a slow variation, which is due to the solution procedure (i.e. iteration towards a steady state).

This leads to the following set of equations, which have been generalised to include internal wave generation, absorbing sponge layers, partial reflection and transmission from breakwaters and other structure, bed friction and wave breaking.

[1] *The phase and group velocity is calculated on basis of linear wave theory.*

$$\lambda_1 \frac{\partial S}{\partial t} + \lambda_2 S + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = S_{Generation}$$

$$\lambda_1 \frac{\partial P}{\partial t} + \lambda_3 P + c_g^2 \frac{\partial S}{\partial x} = 0 \tag{2.4}$$

$$\lambda_1 \frac{\partial Q}{\partial t} + \lambda_3 Q + c_g^2 \frac{\partial S}{\partial y} = 0$$

where

$$\lambda_1 = \frac{c_g}{c}$$

$$\lambda_2 = i \frac{c_g}{c} \omega + f_s \tag{2.5}$$

$$\lambda_3 = (i + f_p) \frac{c_g}{c} \omega + f_s + e_f + e_b$$

Symbol List

ζ	surface elevation above datum, m
P^*	pseudo flux density in the x-direction, $m^3/s/m$
Q^*	pseudo flux density in the y-direction, $m^3/s/m$
S,P,Q	complex function of x, y and t
h	total water depth (=d+ξ), m
c	phase celerity (linear), m/s
c_g	group velocity (linear), m/s
d	still water depth, m
g	gravitational acceleration (= 9.81 m/s^2)
ω	wave frequency, 1/s
i	imaginary unit
$S_{Generation}$	source magnitude per unit horizontal area , $m^3/s/m^2$
f_p	linear friction factor due to energy loss in porous media
f_s	linear friction factor due to energy loss in sponge layers
e_f	energy dissipation due to bed friction
e_b	energy dissipation due wave breaking
x, y	Cartesian co-ordinates, m
t	time, s

2.1.1 Internal Wave Generation

The time-harmonic waves are generated internally inside the model boundaries using a source term in the mass equation. This technique is described by Larsen and Dancy (1983).

Along each generation line, a certain amount of water is added. The added volume of water is determined as $q_{wave}\Delta s\Delta t$, where q_{wave} is the pseudo flux in a progressive wave, Δs the width of the wave front inside a grid mesh and Δt is the time step. The pseudo flux in a progressive wave can be found by assuming a constant form solution to the mild-slope equation. On a horizontal bottom this leads to $q_{wave} = c_g \zeta$, where ζ is the surface elevation of the incoming wave and c_g is the group velocity.

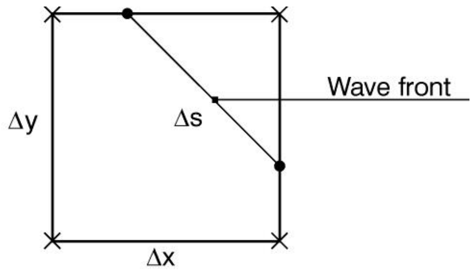

Figure 2.1 Wave front in grid mesh

The source term in Eq. (2.4) can now be determined as the added volume of water divided by the area of the grid mesh and Δt. Hence, considering a time-harmonic input wave with amplitude of unity, we get $S_{Generation} = c_g \frac{\Delta s}{\Delta x \Delta y}$. The added amount of water will propagate in two opposite directions. Hence, only half of the specified wave energy will enter the area of interest. Therefore, two parallel generation lines will be specified in order to obtain an incoming wave height of unity.

As shown by Larsen and Dancy (1983), reflected waves are allowed to cross the generation lines without any distortion or reflection.

2.1.2 Partial Reflection and Transmission

Partial reflection in combination with the mild-slope equations is treated in Madsen and Larsen (1987) on the basis of a porous flow description, which involves the local porosity and a linear dissipation term in the governing equations, Eq. (2.4) and Eq. (2.5). In the following, we shall simplify the original formulation by neglecting the porosity, but including a linear friction term in the momentum equations. In 1D, this leads to the following modification of Eq. (2.2):

$$\frac{\partial P^*}{\partial t} + cc_g \frac{\partial \zeta}{\partial x} + \omega f_p P^* = 0 \tag{2.6}$$

$$\frac{c_g}{c}\frac{\partial \zeta}{\partial t} + \frac{\partial P^*}{\partial x} = 0 \tag{2.7}$$

where f_p is the linear friction factor.

An analysis of Eq. (2.6) and Eq. (2.7) can now easily be made by assuming a constant depth and constant values of f_p, c and c_g:

First of all, P^* can be eliminated from Eq. (2.6) and Eq. (2.7) by the use of cross-differentiation and inserting Eq. (2.3) leads to:

$$\frac{\partial^2 S}{\partial x^2} + \frac{\omega^2}{c^2}\left(1 - if_p\right)S = 0 \tag{2.8}$$

$$P^* = -\frac{cc_g}{\omega\left(1 - if_p\right)}\frac{\partial S}{\partial x} \tag{2.9}$$

Inside a porous media as e.g. a rubble mound, we shall generally look for solutions on the form

$$S(x,t) = a_1 e^{-i\kappa x} + a_2 e^{i\kappa x} \tag{2.10}$$

which in combination with Eq. (2.8) leads to:

$$\kappa \equiv \frac{\omega}{c}\sqrt{1 - if_p} \tag{2.11}$$

Substituting Eq. (2.10) into Eq. (2.9) leads to:

$$P^* \equiv \varepsilon\, c_g\, (a_1 e^{-i\kappa x} + a_2 e^{i\kappa x}) \tag{2.12}$$

where

$$\varepsilon \equiv \frac{1}{\sqrt{1 - if_p}} \tag{2.13}$$

At the front face of the porous structure (at x = 0), we shall now match continuity and pressure, i.e. ζ and P^*. This leads to the conditions

$$\begin{aligned} a_1 + a_2 &= a_i + a_r \\ M\,(a_1 + a_2) &= a_i - a_r \end{aligned} \tag{2.14}$$

where a_i is the incoming wave amplitude, and a_r is the reflected wave amplitude.

Further conditions depend on whether the porous rubble mound has got a permeable or impermeable core. In the first case, Eq. (2.10) and Eq. (2.12) have to be matched with the transmitted wave field and in the latter case, a condition of zero flow has to be applied.

Details can be found in Madsen (1983).

The final analytical expressions become:

$$\frac{a_r}{a_i} = \frac{(1-\varepsilon) + (1+\varepsilon)\, e^{-i2\kappa 2}}{(1+\varepsilon) + (1-\varepsilon)\, e^{-i2\kappa 2}}, \qquad \text{impermeable core}$$

$$\left.\begin{aligned} \frac{a_r}{a_i} &= \frac{(1-\varepsilon^2)(\, e^{i\kappa\kappa} - e^{-i\kappa\kappa}\,)}{(1+\varepsilon)^2\, e^{i\kappa\kappa} - (1-\varepsilon)^2\, e^{-i\kappa\kappa}} \\ \frac{a_t}{a_2} &= \frac{4\,\varepsilon}{(1+\varepsilon)^2\, e^{i\kappa\kappa} - (1-\varepsilon)^2\, e^{-i\kappa\kappa}} \end{aligned}\right\}, \text{ permeable core} \tag{2.15}$$

where *B* is the width of the porous structure and a_t is the amplitude of the transmitted wave.

The reflection/transmission coefficients can now be determined from the above as a function of f_p and *B*, but also of the water depth and the wave period. The solutions can be obtained by using the program Calculation of Reflection Coefficients in the MIKE 21 Toolbox (waves).

2.1.3 Radiation Stresses

The concept of radiation stress was introduced in a series of publications by Longuet-Higgins and Stewart (1960, 1961, 1962 and 1964). They showed the existence of a net momentum flux associated with a progressive wave motion. In regions where waves shoal and break, spatial gradients in the components of the radiation stress cause a net circulation and gradient in the mean-sea-level. The original expressions for the radiation stress given by Longuet-Higgins and Stewart apply to simple harmonic, purely progressive waves and are not correct if any reflected wave is present. A general expression applicable in crossing wave trains was formulated by Bettess and Bettess (1982), and Copeland (1985b) modified this formulation in terms of the variables used in this hyperbolic formulation of the mild-slope equation. The expressions given by Copeland read:

$$\begin{aligned} S_{xx} &= \overline{R_x^2}\, A - \overline{\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y}\right)^2}\, B + \frac{\partial}{\partial x}\left[\overline{R_x\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y}\right) D}\right] \\ &\quad + \frac{\partial}{\partial y}\left[\overline{R_y\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y}\right) D}\right] + \frac{1}{2}\,\rho\, g\, \overline{\zeta^2} \end{aligned} \tag{2.16}$$

$$S_{yy} = S_{xx} + A\left(\overline{R_y^2} - \overline{R_x^2}\right)$$

$$S_{xy} = A\, \overline{R_x\, R_y}$$

where the bar indicates time averaging over one wave period and *A, B,* and *D* are given by:

$$A \equiv \rho\,\kappa\,\frac{1}{4\,sinh^2 kh}(sinh2kh + 2kh)$$

$$B \equiv \frac{\rho}{\kappa}\,\frac{1}{4\,\sinh^2 kh}(\sinh 2kh - 2kh) \tag{2.17}$$

$$D \equiv \rho\,h\,\frac{1}{4\,sinh^2 kh}\left(\frac{sinh2kh}{2kh} - cosh2kh\right)$$

Furthermore, ς is the surface elevation, while R_x and R_y are defined by:

$$R_x \equiv \frac{c}{c_g} P^*(x,y,t)$$

$$R_y \equiv \frac{c}{c_g} Q^*(x,y,t) \tag{2.18}$$

Once again, we utilise the harmonic time variation by expressing:

$$\begin{aligned} \varsigma &= \mathrm{Re}\,\{S\,e^{i\omega t}\} \\ P^* &= \mathrm{Re}\,\{P\,e^{i\omega t}\} \\ Q^* &= \mathrm{Re}\,\{Q\,ei\omega t\} \end{aligned} \tag{2.19}$$

or alternatively

$$\begin{aligned} \varsigma &= SR\cos\omega t - SI\sin\omega t \\ P^* &= PR\cos\omega t - PI\sin\omega t \\ Q^* &= QR\cos\omega t - QI\sin\omega t \end{aligned} \tag{2.20}$$

where the subscripts 'R' and 'I' denote real and imaginary parts respectively.

This leads to the following formulation of the radiation stress valid in crossing wave trains:

$$\begin{aligned} S_{xx} &= \tfrac{1}{2}\gamma^2\;A\left[P_R^2 + P_I^2\right] - \tfrac{1}{2}B\left[F_R^2 + F_I^2\right] \\ &+ \tfrac{1}{2}\frac{\partial}{\partial x}\left[\gamma D P_R F_R + \gamma D P_I F_I\right] \\ &+ \tfrac{1}{2}\frac{\partial}{\partial y}\left[\gamma D Q_R F_R + \gamma D Q_I F_I\right] + \frac{1}{4}\rho\;g\left(S_R^2 + S_I^2\right) \end{aligned} \tag{2.21}$$

$$S_{yy} = S_{xx} + \tfrac{1}{2}\gamma^2\;A\left[\left(Q_R^2 + Q_I^2\right) - \left(P_R^2 + P_I^2\right)\right]$$

$$S_{xy} = S_{yx} = \tfrac{1}{2}\gamma^2\;A\left[P_R Q_R + P_I + Q_I\right]$$

where K, F_R and F_I are defined by

$$F_R \equiv \frac{\partial}{\partial x}(\gamma\, P_R) + \frac{\partial}{\partial y}(\gamma\, Q_R)$$

$$F_I \equiv \frac{\partial}{\partial x}(\gamma\, P_I) + \frac{\partial}{\partial y}(\gamma\, Q_I) \tag{2.22}$$

$$\gamma \equiv \frac{C}{C_g} = \frac{2}{1+G}, \qquad G \equiv \frac{2kh}{sinh2kh}$$

2.2 Numerical Implementation

The numerical method used is based on the same numerical scheme as in MIKE 21 Flow Model, which was originally introduced by Abbott et al (1973).

The differential equations are spatially discretised on a rectangular, staggered grid as illustrated in Figure 2.2. Scalar quantities such as water surface elevation are defined in the grid nodes, whereas flux components are defined halfway between adjacent grid nodes in the respective directions.

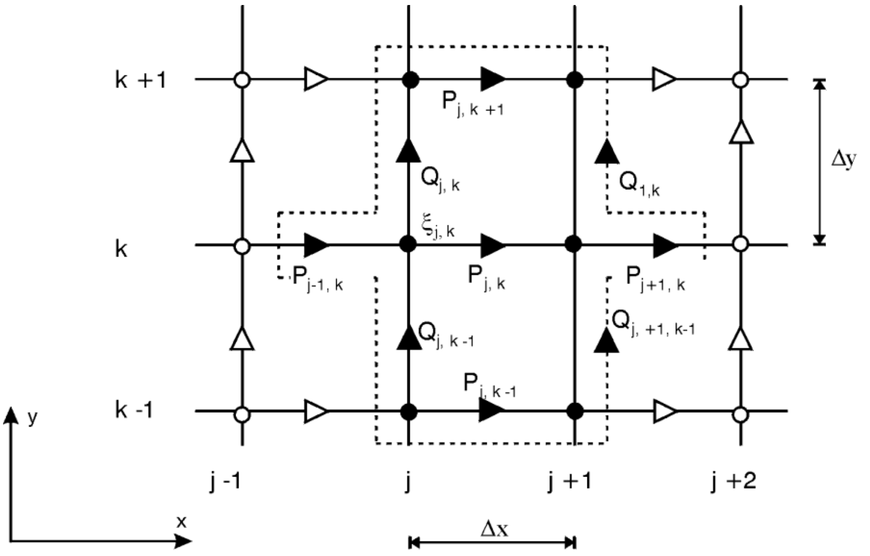

Figure 2.2 Staggered grid in x-y-space

Time centring of the three governing PDE's Eq. (2.4) is achieved by defining ξ at half time levels (i.e. n, n+1/2, n+1, etc.), P at integer time levels (n, n+1, n+2, etc), and Q at half-integer time levels (n+1/2, n+3/2, n+5/2, etc).

Using these discretisations, the three PDE's are formulated as a system of implicit expressions for the unknown values at the grid points, each expression involving known but also unknown values at other grid points and time levels. The finite-difference approximation of the spatial derivatives is a straightforward mid-centring, see details in Appendix A.

The applied algorithm is a non-iterative Alternating Direction Implicit (ADI) algorithm. The resulting tri-diagonal system of equations is solved by the well known Double Sweep Algorithm.

After a certain number of iterations (i.e. time steps), the steady-state solution is obtained and the wave disturbance coefficients or relative wave heights can be determined as the modulus of *S.*

The reader is referred to *Appendix A*, where an in-depth description of the numerical formulation is given.

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APPENDICES

APPENDIX A

In-depth Description of the Numerical Formulation

A In-depth Description of the Numerical Formulation

The present appendix aims at giving an in-depth description of physical, mathematical and numerical background related to elliptic mild-slope wave modelling by inclusion of the following paper:

Madsen, P A & Larsen, J (1987): An efficient finite-difference approach to the mild-slope equation. Coastal Engineering, 11, 329-351.