MIKE 3 FLOW MODEL

Hydrostatic Version

Scientific Documentation

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1 Introduction

This document deals with some of the numerical aspects of the hydrostatic engine, implemented into the MIKE 3 system.

The hydrostatic (HS) version of MIKE 3 HD is much similar to the classical version MIKE 3 HD, which applies an artificial compressibility method (ACM). From a theoretical point of view, the only difference is the hydrostatic assumption and an approximation regarding the vertical velocity. However, from a numerical point of view, the hydrostatic version differs significantly from the ACM version of MIKE 3 HD:

- The numerical scheme is semi-implicit as compared to the ADI-implicit scheme used in the MIKE 3 HD ACM version.
- The artificial compressibility term is inherently excluded.
- The nesting procedure has been re-written.

From a ‘computational speed’ point of view, the hydrostatic version and the ACM version of MIKE 3 HD is about equally fast. Even though the hydrostatic version has one equation less to solve than the standard version, the hydrostatic version deals with both a 3D solution and a 2D depth-integrated solution.

2 Main Equations for MIKE 3 HS

The mathematical foundation for the standard MIKE 3 HD engine is the mass equation and the Reynolds-averaged Navier-Stokes equation, including an artificial compressibility due to the chosen numerical solution procedure. These equations read (only x-direction is shown for 2^{nd} equation):

$$\frac{1}{\rho c_s^2}\frac{\partial P}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=S_{MASS} \tag{2.1}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}+2\omega\left(-v\sin(\phi)+w\cos(\phi)\sin(\lambda)\right)=\\
&-\frac{1}{\rho}\frac{\partial P}{\partial x}\\
&+\frac{\partial}{\partial x}\left(2\upsilon_t\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\upsilon_t\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\upsilon_t\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial z}\right)\right)\\
&+u_{ss}S_{MASS}
\end{aligned} \tag{2.2}$$

where

ρ	density
c_s	speed of sound in water
u,v,w	velocities in x,y,z directions
ω	Coriolis parameter
ϕ,λ	latitude, longitude
υ_t	turbulent eddy viscosity
S_{MASS}	source/sink terms with

$$S_{MASS}=\sum_{i_S=1}^{N_S}\delta\left(x-x_{s,i_S},y-y_{s,i_S},z-z_{s,i_S}\right)Q_{s,i_S}$$

δ	delta function of source/sink coordinates, m^{-3}
$x_{s,i_S},y_{s,i_S},z_{s,i_S}$	coordinates of source/sink No.i_S
Q_{s,i_S}	discharge at source/sink No.i_S , m^3/s

The differences between MIKE 3 HS and MIKE 3 ACM are:

- A hydrostatic pressure assumption is applied, i.e. the vertical accelerations are assumed to be negligible. The vertical velocity w is assumed negligible, resulting in the removal of the secondary Coriolis term and the last diffusion term. The pressure is split up into two parts, the external pressure and the internal pressure.
- The external pressure is directly linked to the free surface, and the internal pressure is due to the density differences.

- The fluid is assumed incompressible, as opposed to the standard version of MIKE 3 HD. Consequently, the compressibility term in the mass equations is discarded.

This simplifies the above equations to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=S_{MASS} \tag{2.3}$$

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}-2\omega v\sin(\phi)=\\&-\frac{1}{\rho}\frac{\partial P}{\partial x}\\&+\frac{\partial}{\partial x}\left(2\upsilon_t\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\upsilon_t\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\upsilon_t\frac{\partial u}{\partial z}\right)\\&+u_{ss}S_{MASS}\end{aligned} \tag{2.4}$$

The external/internal pressure gradient force is given by:

$$\frac{1}{\rho}\frac{\partial P}{\partial x}=g\frac{\rho(\zeta)}{\rho}\frac{\partial \zeta}{\partial x}+\frac{g}{\rho}\int_z^{\zeta}\frac{\partial \rho}{\partial x}dz \tag{2.5}$$

where

g	acceleration due to gravity
ζ	surface elevation

In the ACM version of MIKE 3, the top horizontal layer containing the free surface is solved separately from, but not independently of, the underlying cells. The top layer is layer-integrated as opposed to the underlying cells.

In the hydrostatic version of MIKE 3, the equations to be solved are in their layer-integrated form for both the top layer and the underlying cells. This is due to the solution procedure, where it is convenient to have the same formulation for all cells in each water column. Assuming that the horizontal velocities are constant over the layer thickness, the layer-integrated form of (2.3) - (2.4), with the pressure gradient force inserted, is:

$$\frac{\partial uh}{\partial x}+\frac{\partial vh}{\partial y}+w_{top}-w_{bot}=P-E+\sum_{i_S}\delta\left(x-x_{s,i_S},y-y_{s,i_S}\right)Q_{s,i_S} \tag{2.6}$$

$$
\begin{aligned}
&\frac{\partial uh}{\partial t}+\frac{\partial uuh}{\partial x}+\frac{\partial uvh}{\partial y}+(uw)_{top}-(uw)_{bot}-2\omega vh\sin(\phi)= \\
&-gh\frac{\partial \zeta}{\partial x}\quad-\frac{g}{\rho}\int_{layer}\left(\int_{z'}^{\zeta}\frac{\partial \rho}{\partial x}dz'\right)dz \\
&+\frac{\partial}{\partial x}\left(2\upsilon_t h\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\upsilon_t h\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\left(\upsilon_t\frac{\partial u}{\partial z}\right)_{top}-\left(\upsilon_t\frac{\partial u}{\partial z}\right)_{bot} \\
&+u_{ss}\sum_{i_S}\delta\left(x-x_{s,i_S},y-y_{s,i_S}\right)Q_{s,i_S}
\end{aligned}
\tag{2.7}
$$

where the sums represent all point sources/sinks in the considered layer, and precipitation and evaporation terms, *P* and *E* (m/s), have been excluded from the sum. The precipitation and evaporation terms is only included if the considered layer is the surface layer.

The depth-integrated version of (2.6) is:

$$
\frac{\partial \zeta}{\partial t}+\frac{\partial UH}{\partial x}+\frac{\partial VH}{\partial y}=P-E+\sum_{i_S}\delta\left(x-x_{s,i_S},y-y_{s,i_S}\right)Q_{s,i_S}
\tag{2.8}
$$

with sum over all point sources/sinks.

3 Solution Method

Both the ACM and HS versions of MIKE 3 HD apply the same grid (Akawara C), and uses time staggering. The ACM version applies three sub time steps in order to advance one time step, one sub time step for each spatial direction, see Figure 3.1. The HS version only contains two momentum equations to solve, and thus only needs two sub time steps for advancing one time step.

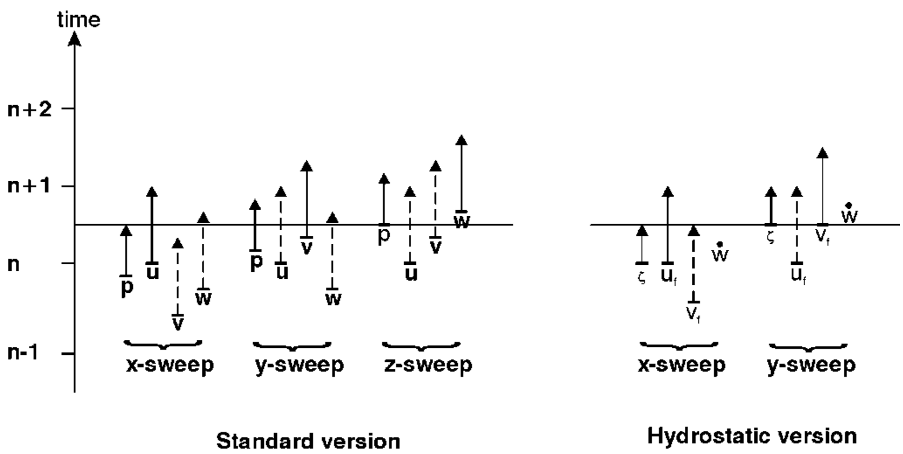

Figure 3.1 Time centring

Due to the ADI solution technique, the ACM version of MIKE 3 HD is time centred and fully implicit in the ADI sense.

The hydrostatic version does also apply the line sweeps as in the ADI solution technique, but does not make use of the ADI side-feeding, and is furthermore a semi-implicit model.

The basic idea of the solution procedure for the hydrostatic version is to work with both the depth-integrated and layer-integrated version of the continuity and momentum equations. This resembles somewhat models like the Princeton Ocean Model (POM), where the computations are split up into an external and internal mode. Where POM use a different time step for the external and internal mode, the hydrostatic version of MIKE 3 HD solves for both the external and internal mode at each time step, thus keeping full consistency between the two modes.

3.1 Solution procedure

A five-step procedure is followed when solving the depth-integrated and layer-integrated continuity equation and momentum equation for each direction. In the following, this procedure is outlined for the x-direction. For convenience, the layer-integrated and the depth-integrated horizontal flux density (uh) respectively (UH) is written as u_f respectively U_f, and similar for the y-direction.

First, the coefficients for the layer-integrated momentum equation are calculated for each column of water. The discretized equation is written as:

$$\left. \begin{array}{l} AL^*_{j,l} \cdot \varsigma_j^{n+½} + DOL^*_{j,l} \cdot u_{f\ j,l-1}^{\ n+1} + BL^*_{j,l} \cdot u_{f\ j,l}^{\ n+1} \\ + UPL^*_{j,l} \cdot u_{f\ j,l+1}^{\ n+1} + CL^*_{j,l} \cdot \varsigma_{j+1}^{n+½} = DL^*_{j,l} \end{array} \right|_k \tag{3.1}$$

A partial elimination is performed on (3.1) in order reduce the equation to:

$$AL^{\bullet}_{j,l} \cdot \varsigma_j^{n+½} + u_{f\ j,l}^{\ n+1} + CL^{\bullet}_{j,l} \cdot \varsigma_{j+1}^{n+½} = DL^{\bullet}_{j,l} \Big|_k \tag{3.2}$$

Second, a summation of the reduced, layer-integrated momentum equation is performed for each water column over all layers:

$$\sum_{Layers} (AL^{\bullet}_{j,l} \cdot \varsigma_j^{n+½}) + \sum_{Layers} u_{f\ j,l}^{\ n+1} + \sum_{Layers} (CL^{\bullet}_{j,l} \cdot \varsigma_{j+1}^{n+½}) = \sum_{Layers} (DL^{\bullet}_{j,l}) \Big|_k \tag{3.3}$$

Since the surface elevation ζ is independent of each layer, (3.3) can also be written as:

$$\left(\sum_{Layers} AL^{\bullet}_{j,l} \right) \cdot \varsigma_j^{n+½} + \sum_{Layers} u_{f\ j,l}^{\ n+1} + \left(\sum_{Layers} CL^{\bullet}_{j,l} \right) \cdot \varsigma_{j+1}^{n+½} = \sum_{Layers} (DL^{\bullet}_{j,l}) \Big|_k \tag{3.4}$$

Setting:

$$U_{f\ j}^{\ n+1} = \sum_{Layers} u_{f\ j,l}^{\ n+1}$$

$$A^{\bullet}_j = \sum_{Layers} AL^{\bullet}_{j,l}$$

$$B^{\bullet}_j = 1$$

$$C^{\bullet}_j = \sum_{Layers} CL^{\bullet}_{j,l}$$

$$D^{\bullet}_j = \sum_{Layers} DL^{\bullet}_{j,l}$$

gives the final form of the summed layer-integrated momentum equations:

$$A^{\bullet}_j \cdot \varsigma_j^{n+½} + B^{\bullet}_j \cdot U_{f\ j}^{\ n+1} + C^{\bullet}_j \cdot \varsigma_{j+1}^{n+½} = D^{\bullet}_j \Big|_k \tag{3.5}$$

Third, the coefficients corresponding to the discretized depth-integrated continuity equation are calculated:

$$A_j \cdot U_{f\ j-1}^{\ n+1} + B_j \cdot \varsigma_j^{n+½} + C_j \cdot U_{f\ j}^{\ n+1} = D_j \Big|_k \tag{3.6}$$

Fourth, (3.5) and (3.6) is solved using the well-known double-sweep algorithm, with appropriate boundary conditions inserted. The solution obtained is the depth-integrated flux density UH and the surface elevation ς along the line. Knowing the surface elevation, the layer-integrated flux density can easily be obtained by rearranging (3.2):

$$u_{f\ j,l}^{\ n+1} = DL^{\bullet}_{j,l} - AL^{\bullet}_{j,l} \cdot \varsigma_j^{n+½} - CL^{\bullet}_{j,l} \cdot \varsigma_{j+1}^{n+½}\Big|_k \tag{3.7}$$

Fifth, the layer-integrated continuity equation (2.6) is used to calculate the vertical velocity. It is solved explicitly, looping from the bottom to the surface.

4 Discretization

The discretization of the equations is shown only for the x-direction, since the discretized equations in the y-direction are analogous.

The notation for coefficient contributions is 'C-like', i.e.

$$A \quad += \quad "term"$$

means that the '*term*' on the right hand side of $+=$ is added to coefficient *A*.

4.1 Continuity equation

4.1.1 Layer-integrated continuity equation

The layer-integrated continuity equation is discretized as:

$$\begin{aligned}
&\frac{1}{\Delta x}\frac{1}{2}\left\{\left(u_{f_{j,k,l}} - u_{f_{j-1,k,l}}\right)^{n} + \left(u_{f_{j,k,l}} - u_{f_{j-1,k,l}}\right)^{n+1}\right\} \\
&+\frac{1}{\Delta y}\frac{1}{2}\left\{\left(v_{f_{j,k,l}} - v_{f_{j,k-1,l}}\right)^{n-½} + \left(v_{f_{j,k,l}} - v_{f_{j,k-1,l}}\right)^{n+½}\right\} \\
&+ w_{top_{j,k,l}}^{n+1/4} - w_{bot_{j,k,l}}^{n+1/4} = P_{j,k}^{n+1} - E_{j,k}^{n+1} + \sum_{j,k}\frac{Q_{s,i_S}^{n+1}}{\Delta x \Delta y}
\end{aligned} \tag{4.1}$$

where the sum is over all point sources/sinks in the considered grid cell at (*j*,*k*), and *P* and *E* are only included if the considered grid cell is in the surface layer.

As stated earlier, the layer-integrated continuity equation is used for calculating the vertical velocity. This is simply done explicitly, using the boundary condition $w_{bot_{j,k,l}}^{n+1/4} = 0$ at the lowest box.

4.1.2 Depth-integrated continuity equation

The depth-integrated continuity equation is likewise discretized as:

$$\begin{aligned}
&\frac{\zeta_{j,k}^{n+½} - \zeta_{j,k}^{n}}{\frac{1}{2}\Delta t} + \frac{1}{\Delta x}\frac{1}{2}\left\{\left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^{n} + \left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^{n+1}\right\} \\
&\qquad + \frac{1}{\Delta y}\frac{1}{2}\left\{\left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n-½} + \left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n+½}\right\} \\
&\qquad = P_{j,k}^{n+1} - E_{j,k}^{n+1} + \sum_{j,k}\frac{Q_{s,i_S}^{n+1}}{\Delta x \Delta y}
\end{aligned} \tag{4.2}$$

where the sum is over all point sources/sinks in the considered column at (*j*,*k*).

Coefficient contribution

$$A_j += -\frac{1}{2}\frac{1}{\Delta x}$$

$$B_j += 2\frac{1}{\Delta t}$$

$$C_j += \frac{1}{2}\frac{1}{\Delta x}$$

$$\begin{aligned} D_j += {} & 2\frac{1}{\Delta t}\zeta^n_{j,k} - \frac{1}{2}\frac{1}{\Delta x}\left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^n \\ & -\frac{1}{2}\frac{1}{\Delta y}\left\{\left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n-½} + \left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n+½}\right\} \\ & + P^{n+1}_{j,k} - E^{n+1}_{j,k} + \sum_{j,k}\frac{Q^{n+1}_{s,i_S}}{\Delta x \Delta y} \end{aligned} \tag{4.3}$$

4.2 Momentum equation

The various terms in the momentum equation are in the following developed one by one.

4.2.1 The time derivative term

The straightforward approximation to the time derivative term is

$$\frac{\partial uh}{\partial t} \approx \frac{u^{n+1}_{f_{j,k,l}} - u^n_{f_{j,k,l}}}{\Delta t}$$

Coefficient contribution

$$BL^{\bullet} += \frac{1}{\Delta t}$$

$$DL^{\bullet} += \frac{1}{\Delta t}u^n_{f_{j,k,l}} = \frac{1}{\Delta t}u^n_{j,k,l}\,\tfrac{1}{2}\left(h^n_{j,k,l} + h^n_{j+1,k,l}\right) \tag{4.4}$$

4.2.2 The gravity term

The approximation to the gravity term is

$$gh\frac{\partial\zeta}{\partial x} \approx g\,\tfrac{1}{2}\left(h_{j,k,l}^{n} + h_{j+1,k,l}^{n}\right)\frac{\zeta_{f_{j+1,k}}^{n+½} - \zeta_{f_{j,k}}^{n+½}}{\Delta x}$$

Coefficient contribution

$$AL^{\bullet} += -\frac{1}{\Delta x}\,g\,\tfrac{1}{2}\left(h_{j,k,l}^{n} + h_{j+1,k,l}^{n}\right)$$
$$CL^{\bullet} += \frac{1}{\Delta x}\,g\,\tfrac{1}{2}\left(h_{j,k,l}^{n} + h_{j+1,k,l}^{n}\right) \tag{4.5}$$

4.2.3 The buoyancy term

The layer-integrated buoyancy term can be expressed as

$$\int_{layer}\left(\frac{g}{\rho}\int_{z'}^{\zeta}\frac{\partial\rho}{\partial x}dz'\right)dz = \frac{gh}{\bar{\rho}}\int_{h^+}^{\zeta}\frac{\partial\rho}{\partial x}dz + \frac{1}{\bar{\rho}}\int_{h^-}^{h^+}\left(z-h^-\right)\frac{\partial\rho}{\partial x}dz \tag{4.6}$$

where $h=h^+ - h^-$ is the layer thickness and $\bar{\rho}$ is the mean density at the specific location, i.e.

$$h_l \approx \tfrac{1}{2}\left(h_{j,k,l}^{n} + h_{j+1,k,l}^{n}\right)$$
$$\bar{\rho} \approx \tfrac{1}{2}\left(\rho_{j,k,l}^{n} + \rho_{j+1,k,l}^{n}\right)$$

The first integral is simply approximated by a sum of density gradients in layers above the present layer *l*:

$$\int_{h^+}^{\zeta}\frac{\partial\rho}{\partial x}dz \approx \sum_{L=l+1}^{lextr}\frac{\rho_{j+1,k,l} - \rho_{j,k,l}}{\Delta x}h_L$$

The second integral is approximated by assuming the density gradient is constant over the layer, and then calculating the resulting integral analytically:

$$\int_{h^-}^{h^+}\left(z-h^-\right)\frac{\partial\rho}{\partial x}dz \approx \frac{\rho_{j+1,k,l} - \rho_{j,k,l}}{\Delta x}\int_{h^-}^{h^+}\left(z-h^-\right)dz = \frac{\rho_{j+1,k,l} - \rho_{j,k,l}}{\Delta x}\frac{1}{2}h_l^2$$

The coefficient contribution becomes:

$$DL^{\bullet} += -\frac{gh_l^2}{2\bar{\rho}}\frac{\rho_{j+1,k,l}-\rho_{j,k,l}}{\Delta x} - \frac{gh_l}{\bar{\rho}}\sum_{L=l+1}^{lextr}\frac{\rho_{j+1,k,L}-\rho_{j,k,L}}{\Delta x}h_L \tag{4.7}$$

4.2.4 The Coriolis term

The straightforward approximation to the Coriolis term is

$$2\omega\sin(\phi)vh \approx 2\omega\sin(\phi)\tfrac{1}{4}\left(v_{j,k,l}^{n+½}+v_{j+1,k,l}^{n+½}+v_{j,k-1,l}^{n+½}+v_{j+1,k-1,l}^{n+½}\right)\tfrac{1}{2}\left(h_{j,k,l}^{n}+h_{j+1,k,l}^{n}\right)$$

Coefficient contribution

$$DL^{\bullet} += 2\omega\sin(\phi)\tfrac{1}{4}\left(v_{j,k,l}^{n+½}+v_{j+1,k,l}^{n+½}+v_{j,k-1,l}^{n+½}+v_{j+1,k-1,l}^{n+½}\right)\tfrac{1}{2}\left(h_{j,k,l}^{n}+h_{j+1,k,l}^{n}\right) \tag{4.8}$$

4.2.5 The source/sink term

The implementation of the source/sink term is

$$u_{SS}\sum_{i_S}\frac{Q_{s,i_S}}{\Delta x\Delta y} \approx u_{SS}^{n+1}\sum_{j,k}\frac{Q_{s,i_S}^{n+1}}{\Delta x\Delta y}$$

where the sum is over over all sources/sinks in the considered grid cell at (*j*,*k*).

Coefficient contribution

$$DL^{\bullet} += u_{SS}^{n+1}\sum_{j,k}\frac{Q_{s,i_S}^{n+1}}{\Delta x\Delta y} \tag{4.9}$$

4.2.6 The shear terms

The three shear terms are developed using central differences. Due to the solution procedure, the two horizontal shear terms are discretized explicitly. The first horizontal shear term is discretized as

$$\frac{\partial}{\partial x}\left(2\upsilon_t h\frac{\partial u}{\partial x}\right) \approx \frac{1}{\Delta x}\left\{2\upsilon_{t\,j+1,k,1}h_{j+1,k,l}^{n}\frac{u_{j+1,k,l}^{n}-u_{j,k,l}^{n}}{\Delta x} - 2\upsilon_{t\,j,k,1}h_{j,k,l}^{n}\frac{u_{j,k,l}^{n}-u_{j-1,k,l}^{n}}{\Delta x}\right\}$$

Coefficient contribution

$$DL^{\bullet} += 2\frac{1}{\Delta x^2}\left\{\upsilon_{t\,j+1,k,1}h_{j+1,k,l}^{n}\left(u_{j+1,k,l}^{n}-u_{j,k,l}^{n}\right)-\upsilon_{t\,j,k,1}h_{j,k,l}^{n}\left(u_{j,k,l}^{n}-u_{j-1,k,l}^{n}\right)\right\} \tag{4.10}$$

The second horizontal shear term is discretized as

$$\frac{\partial}{\partial y}\left(\upsilon_t h\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)\approx$$

$$\frac{1}{\Delta y}\left\{\upsilon_{tL}h_L\left(\frac{u^n_{j,k+1,l}-u^n_{j,k,l}}{\Delta y}+\frac{v^{n+½}_{j+1,k,l}-v^{n+½}_{j,k,l}}{\Delta x}\right)\right.$$
$$\left.-\upsilon_{tR}h_R\left(\frac{u^n_{j,k,l}-u^n_{j,k-1,l}}{\Delta y}+\frac{v^{n+½}_{j+1,k-1,l}-v^{n+½}_{j,k-1,l}}{\Delta x}\right)\right\}$$

using the abbreviations

$$h_L=\tfrac{1}{4}\left(h^n_{j,k,l}+h^n_{j+1,k,l}+h^n_{j,k+1,l}+h^n_{j+1,k+1,l}\right)$$

$$h_R=\tfrac{1}{4}\left(h^n_{j,k-1,l}+h^n_{j+1,k-1,l}+h^n_{j,k,l}+h^n_{j+1,k,l}\right)$$

$$\upsilon_{tL}=\tfrac{1}{4}\left(\upsilon_{t\,j,k,l}+\upsilon_{t\,j+1,k,l}+\upsilon_{t\,j,k+1,l}+\upsilon_{t\,j+1,k+1,l}\right) \tag{4.11}$$

$$\upsilon_{tL}=\tfrac{1}{4}\left(\upsilon_{t\,j,k-1,l}+\upsilon_{t\,j+1,k-1,l}+\upsilon_{t\,j,k,l}+\upsilon_{t\,j+1,k,l}\right)$$

Coefficient contribution

$$DL^{\bullet}+=\frac{1}{\Delta y}\left\{\upsilon_{tL}h_L\left(\frac{u^n_{j,k+1,l}-u^n_{j,k,l}}{\Delta y}+\frac{v^{n+½}_{j+1,k,l}-v^{n+½}_{j,k,l}}{\Delta x}\right)\right.$$
$$\left.-\upsilon_{tR}h_R\left(\frac{u^n_{j,k,l}-u^n_{j,k-1,l}}{\Delta y}+\frac{v^{n+½}_{j+1,k-1,l}-v^{n+½}_{j,k-1,l}}{\Delta x}\right)\right\} \tag{4.12}$$

The third shear stress term accounts for the shear stress between each layer, and is discretized implicitly, as

$$\left(\upsilon_t\frac{\partial u}{\partial z}\right)_{top}-\left(\upsilon_t\frac{\partial u}{\partial z}\right)_{bot}\approx$$

$$\upsilon_{t\,top}\frac{\frac{1}{2}\left(u^n_{j,k,l+1}+\left(\frac{u^{n+1}_f}{h^n_x}\right)_{j,k,l+1}\right)-\frac{1}{2}\left(u^n_{j,k,l}+\left(\frac{u^{n+1}_f}{h^n_x}\right)_{j,k,l}\right)}{\Delta z}$$

$$-\upsilon_{t\,bot}\frac{\frac{1}{2}\left(u^n_{j,k,l}+\left(\frac{u^{n+1}_f}{h^n_x}\right)_{j,k,l}\right)-\frac{1}{2}\left(u^n_{j,k,l-1}+\left(\frac{u^{n+1}_f}{h^n_x}\right)_{j,k,l-1}\right)}{\Delta z}$$

using the abbreviations

$$\upsilon_{t\,top} = \tfrac{1}{4}\left(\upsilon_{t\,j,k,l} + \upsilon_{t\,j+1,k,l} + \upsilon_{t\,j,k,l+1} + \upsilon_{t\,j+1,k,l+1}\right)$$
$$\upsilon_{t\,bot} = \tfrac{1}{4}\left(\upsilon_{t\,j,k,l-1} + \upsilon_{t\,j+1,k,l-1} + \upsilon_{t\,j,k,l} + \upsilon_{t\,j+1,k,l}\right) \tag{4.13}$$
$$h^{n}_{x\;j,k,l} = \tfrac{1}{2}\left(h^{n}_{j,k,l} + h^{n}_{j+1,k,l}\right)$$

Coefficient contribution

$$DOL^{\bullet} += -\frac{1}{2}\upsilon_{t\,bot}\frac{1}{h^{n}_{x\;j,k,l-1}\Delta z}$$
$$BL^{\bullet} += \frac{1}{2}\upsilon_{t\,top}\frac{1}{h^{n}_{x\;j,k,l}\Delta z} \quad + \quad \frac{1}{2}\upsilon_{t\,bot}\frac{1}{h^{n}_{x\;j,k,l}\Delta z}$$
$$UPL^{\bullet} += -\frac{1}{2}\upsilon_{t\,top}\frac{1}{h^{n}_{x\;j,k,l+1}\Delta z} \tag{4.14}$$
$$DL^{\bullet} += \frac{1}{2}\upsilon_{t\,top}\frac{1}{\Delta z}\left(u^{n}_{j,k,l+1} - u^{n}_{j,k,l}\right) - \frac{1}{2}\upsilon_{t\,bot}\frac{1}{\Delta z}\left(u^{n}_{j,k,l} - u^{n}_{j,k,l-1}\right)$$

4.2.7 The advective terms

The advective terms in the momentum equation read

$$\frac{\partial uuh}{\partial x} + \frac{\partial uvh}{\partial y} + (uw)_{top} - (uw)_{bot}$$

The two first advective terms needs to be explicit due to the solution algorithm. The third advective term is also implemented explicitly in order to keep all three terms discretized similarly, although it could be implemented implicitly within the solution algorithm.

Three different implementations exist for the advective terms. The first is the non-diffusive central difference scheme, which is second-order accurate in space. The negative effect of the central difference scheme used on the advective terms is wiggling, caused by the non-directional nature of the scheme. The second implementation is a simple upwind scheme, putting the weight more on the upwind part when the flow is more directional. The advantage is the avoidance of wiggling, and the disadvantage is of course the diffusive effect of the upwind part of the scheme. The third and last implementation is the quick scheme, having the advantage of upwind schemes to avoid wiggling without being overly diffusive, and the disadvantage of being slightly more computationally expensive.

At present, only the upwind scheme has been satisfacory tested and is therefore the only recommended scheme.

Central discretization of the advective terms

The central discretization of the central terms reads

$$
\begin{aligned}
&\frac{\partial uuh}{\partial x}+\frac{\partial uvh}{\partial y}+(uw)_{top}-(uw)_{bot}\approx \\
&\frac{\frac{1}{4}\left(u^n_{j,k,l}+u^n_{j+1,k,l}\right)^2 h^n_{j+1,k,l}-\frac{1}{4}\left(u^n_{j-1,k,l}+u^n_{j,k,l}\right)^2 h^n_{j,k,l}}{\Delta x}+ \\
&\frac{h_L\,\frac{1}{2}\left(u^n_{j,k,l}+u^n_{j,k+1,l}\right)\frac{1}{2}\left(v^{n+½}_{j,k,l}+v^{n+½}_{j+1,k,l}\right)-h_R\,\frac{1}{2}\left(u^n_{j,k-1,l}+u^n_{j,k,l}\right)\frac{1}{2}\left(v^{n+½}_{j,k-1,l}+v^{n+½}_{j+1,k-1,l}\right)}{\Delta y} \\
&+\frac{1}{2}\left(u^n_{j,k,l}+u^n_{j,k,l+1}\right)\frac{1}{2}\left(w^n_{j,k,l}+w^n_{j+1,k,l}\right)-\frac{1}{2}\left(u^n_{j,k,l-1}+u^n_{j,k,l}\right)\frac{1}{2}\left(w^n_{j,k,l-1}+w^n_{j+1,k,l-1}\right)
\end{aligned}
$$

using the abbreviations

$$
\begin{aligned}
h_L&=\frac{1}{4}\left(h^n_{j,k,l}+h^n_{j+1,k,l}+h^n_{j,k+1,l}+h^n_{j+1,k+1,l}\right)\\
h_R&=\frac{1}{4}\left(h^n_{j,k-1,l}+h^n_{j+1,k-1,l}+h^n_{j,k,l}+h^n_{j+1,k,l}\right)
\end{aligned}
\tag{4.15}
$$

The so-called Abbott's terms are also implemented in an explicit manner; see the standard MIKE 3 manual for the derivations of these terms.

Coefficient contribution

$$
\begin{aligned}
&DOL^{\bullet}+= \\
&-\frac{1}{2}\frac{1}{\Delta x}\left(\left(u^n_{j,k,l}+u^n_{j+1,k,l}\right)^2 h^n_{j+1,k,l}-\left(u^n_{j-1,k,l}+u^n_{j,k,l}\right)^2 h^n_{j,k,l}\right)\\
&-\frac{1}{4}\frac{1}{\Delta y}\left(h_L\left(u^n_{j,k,l}+u^n_{j,k+1,l}\right)\left(v^{n+½}_{j,k,l}+v^{n+½}_{j+1,k,l}\right)-h_R\left(u^n_{j,k-1,l}+u^n_{j,k,l}\right)\left(v^{n+½}_{j,k-1,l}+v^{n+½}_{j+1,k-1,l}\right)\right)\\
&-\frac{1}{4}\left(\left(u^n_{j,k,l}+u^n_{j,k,l+1}\right)\left(w^n_{j,k,l}+w^n_{j+1,k,l}\right)-\left(u^n_{j,k,l-1}+u^n_{j,k,l}\right)\left(w^n_{j,k,l-1}+w^n_{j+1,k,l-1}\right)\right)
\end{aligned}
\tag{4.16}
$$

Simple upwind discretization of the advective terms

As the discretization technique is similar for all three advective terms, only the discretization of the first term is shown.

Depending upon the sign of the old velocity $u_{R,old}$ at the right cell interface (for the momentum control box, the estimated velocity $u_{R,est}$ to be used in the discretization is given by

$$u_{R,old} = \frac{1}{2}\left(u^n_{j,k,l} + u^n_{j+1,k,l}\right) \geq 0: \quad u_{R,est} = u^n_{j,k,l}$$

or (4.17)

$$u_{R,old} = \frac{1}{2}\left(u^n_{j,k,l} + u^n_{j+1,k,l}\right) < 0: \quad u_{R,est} = u^n_{j+1,k,l}$$

The estimated velocity $u_{L,est}$ at the left cell interface is calculated similarly.

Coefficient contribution

$$DOL^{\bullet} += -\frac{1}{\Delta x}\left(u_{R,old}u_{R,est}h^n_{j+1,k,l} - u_{L,old}u_{L,est}h^n_{j,k,l}\right) \tag{4.18}$$

Quick discretization of the advective terms

As the discretization technique again is similar for all three advective terms, only the discretization of the first term is shown.

As for the hybrid scheme, the velocity at the cell interfaces again need to be estimated from the old velocities. The standard quick scheme gives

$$u_{R,old} = \frac{1}{2}\left(u^n_{j,k,l} + u^n_{j+1,k,l}\right) \geq 0: \quad u_{R,est} = -\frac{1}{6}u^n_{j-1,k,l} + \frac{5}{6}u^n_{j,k,l} + \frac{2}{6}u^n_{j+1,k,l}$$

or (4.19)

$$u_{R,old} = \frac{1}{2}\left(u^n_{j,k,l} + u^n_{j+1,k,l}\right) \leq 0: \quad u_{R,est} = -\frac{1}{6}u^n_{j+2,k,l} + \frac{5}{6}u^n_{j+1,k,l} + \frac{2}{6}u^n_{j,k,l}$$

The estimated velocity $u_{L,est}$ at the left cell interface is calculated similarly. Near land, the implementation switches to simple upstream if the needed values are at land.

Coefficient contribution

$$DOL^{\bullet} += -\frac{1}{\Delta x}\left(u_{R,est}{}^2 h^n_{j+1,k,l} - u_{L,est}{}^2 h^n_{j,k,l}\right) \tag{4.20}$$

5 Internal Coarse-Fine Grid Border

The treatment of the internal coarse-fine grid border has been changed significantly from the classic ACM version to the hydrostatic version of MIKE 3 HD.

In the ACM version, the coefficients for the discretized continuity equation in a common pressure point are calculated in both the coarse grid and the fine grid. The connection between the coarse grid and fine grid is made when setting up the matrix for the double sweep, by eliminating common variables. The unconnected lines in the finer grid apply interpolated/extrapolated values as boundary conditions. It is mostly the determination of the boundary conditions for the unconnected lines causing the mass inaccuracies, since these cannot be calculated in a mass-conservative way.

In order to avoid this problem in the hydrostatic version, the coarse-fine grid border is treated using a finite volume discretization. Thus, the continuity equation for a common elevation point is not discretized in both the coarse grid and fine grid, but rather discretized in a box, which extends to location of the flux points.

In Figure 5.1, the connected lines are marked with red lines and the unconnected lines with green lines.

Using a finite volume discretization, the discretized, depth-integrated continuity equation for BOX 1 in Figure 5.1 reads

$$
\begin{aligned}
&\left(\zeta_{j,k}^{n+½} - \zeta_{j,k}^{n}\right) 2\Delta y 2\Delta x \\
&+ \frac{1}{2}\left\{\left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^{n} + \left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^{n+1}\right\} 2\Delta y \tfrac{1}{2}\Delta t \\
&+ \frac{1}{2}\left\{\left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n-½} + \left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n+½}\right\} 2\Delta x \tfrac{1}{2}\Delta t = 0
\end{aligned}
\qquad (5.1)
$$

where Δx and Δy belongs to the fine grid. Likewise, the discretized, depth-integrated, continuity equation for BOX 2 reads

$$
\begin{aligned}
&\left(\zeta_{j,k}^{n+½} - \zeta_{j,k}^{n}\right) \Delta y 2\Delta x + \frac{1}{2}\left\{\left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^{n} + \left(U_{f_{j,k}} - U_{f_{j-1,k}}\right)^{n+1}\right\} \Delta y \tfrac{1}{2}\Delta t \\
&\quad + \frac{1}{2}\left\{\left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n-½} + \left(V_{f_{j,k}} - V_{f_{j,k-1}}\right)^{n+½}\right\} 2\Delta x \tfrac{1}{2}\Delta t = 0
\end{aligned}
\qquad (5.2)
$$

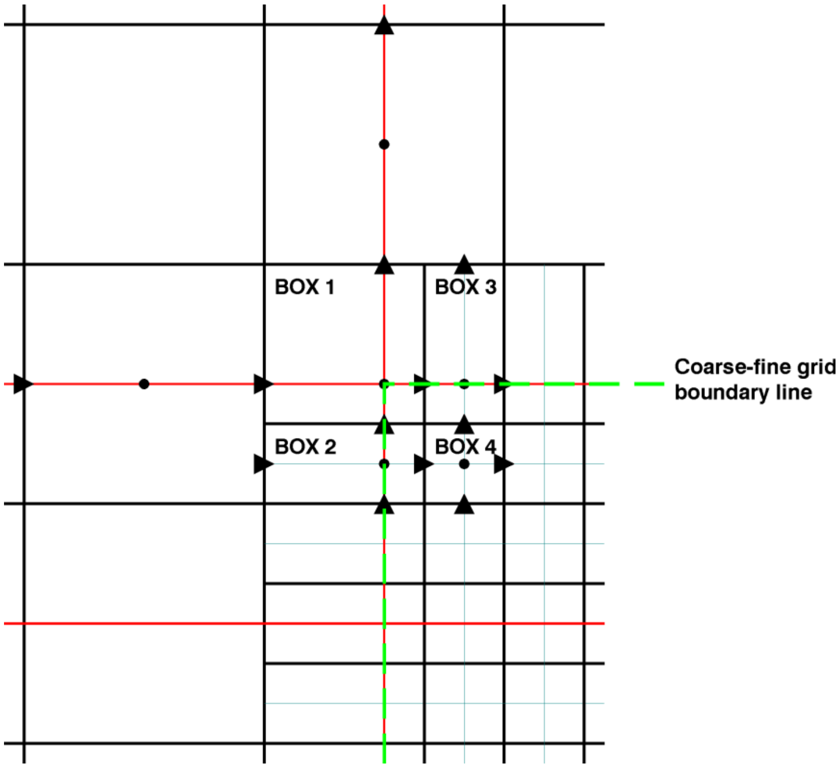

Figure 5.1 Coarse-fine grid connection (connected lines in red, unconnected lines in green)

Using this approach makes it necessary to solve the connected line at the coarse-fine grid boundary line in the fine grid, as opposed to the coarse grid in the ACM version of MIKE 3 HD.

It is noticed that all connected lines must be solved before the unconnected lines are solved. This is due to the fact that the unconnected lines need a boundary condition, which stems from the nearest connected line. Example: The flux at the left cell face in BOX 1 represents the flux for the right cell face in the coarse cell left of BOX 1, and thus also represents the flux needed as a boundary condition for BOX 2. Thus, when the connected line passing through BOX 1 and BOX 3 has been solved, $U_{f\,j-1,k}^{\,n+1}$ is known and applied as a boundary condition when setting up the continuity equation for BOX 2 as shown above. This approach ensures that the flux represented in the coarse grid at the right cell face is exactly equal to the sum of the fluxes at the left cell face of BOX 1 and BOX 2.

6 References

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