MIKE
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MIKE 21

Parabolic Mild-Slope Wave Module

Scientific Documentation

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CONTENTS

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1 Introduction

The present document aims at giving a description of the equations and numerical formulations used in the Parabolic Mild-Slope Module of MIKE 21, MIKE 21 PMS.

First, the basic equations are described. This is followed by a number of sections describing parabolic approximations, introduction of wave dissipation, method of superposition (used for simulating irregular and/or directional waves) and the numerical solution method.

2 Basic Equation

MIKE 21 PMS is based on a parabolic approximation to the elliptic mild-slope equation, which is the governing equation for description of refraction, diffraction and reflection of linear time harmonic water waves on a gently sloping bottom. The equation was first derived by Berkhoff (1972).

The elliptic mild-slope equation can be written as:

$$\nabla \cdot \left(CC_g \nabla \phi \right) + \left(k^2 CC_g + i\omega W \right) \phi = 0 \tag{2.1}$$

where

∇	two-dimensional gradient operator, $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$
C (x,y)	phase speed
Cg (x,y)	group velocity
ϕ (x,y)	mean free surface velocity potential, related to the velocity potential ϕ as

$$\phi(x,y,z,t) = \frac{g}{\omega} \phi(x,y) \frac{\operatorname{Cosh} k(z+d)}{\operatorname{Cosh} kd} e^{-i\omega t} \tag{2.2}$$

z	water level elevation measured from mean water level upwards
d	water depth
k	wave number = 2π/L
W	dissipation term = Ediss/E
Ediss	mean energy dissipation rate per unit time per unit area
E	mean energy per unit area
ω	circular frequency = 2πf
L	wave length
f	frequency

Note also that the free surface elevation, η can be written as (Dean & Dalrymple, 1984):

$$\eta = \frac{1}{g} \frac{\partial \phi}{\partial t} \Big|_{z=0} \tag{2.3}$$

$$\eta = \phi(x,y) e^{-i(\omega t + \pi/2)}$$

For plane progressive waves, the mean free surface potential can be written as:

$$\phi = A^*(x,y) e^{i\psi} \tag{2.4}$$

where

$$\psi = \int^x k\cos\theta dx + \int^y k\sin\theta dy \tag{2.5}$$

and θ is the angle between the wave propagation direction and the x-axis. Now, assuming a predominant wave direction along the x-axis, the phase function ψ can be written as:

$$\psi = \int^{x} k dx \tag{2.6}$$

A parabolic approximation to Eq. (2.1) is obtained by assuming a predominant wave direction, the x-direction, and neglecting back-scatter and diffraction along this direction.

Eq. (2.1) can be expanded as:

$$(CC_g \phi_x)_x + (CC_g \phi_y)_y + (k^2 CC_g + i\omega W)\phi = 0 \tag{2.7}$$

where the subscripts x,y imply derivative with respect to x and y, respectively.

Using Eq. (2.4), the gradient terms can be expressed as:

$$\phi_x = (ikA^* + A_x^*) e^{i\psi} \tag{2.8}$$

$$CC_g \phi_x = (i\omega C_g A^* + CC_g A_x^*) e^{i\psi} \tag{2.9}$$

$$\begin{aligned}(CC_g \phi_x)_x &= \left[ik\left(i\omega C_g A^* + CC_g A_x^*\right)\right. \\ &\left. + i\omega C_g A_x^* + i\omega A^* (C_g)_x + (CC_g A_x^*)_x\right] e^{i\psi}\end{aligned} \tag{2.10}$$

The last term in Eq. (2.10), $(CC_g A^*_x)_x$ representing the influence of back scatter and diffraction along the x-direction is neglected in the parabolic approximation. Thus,

$$(CC_g \phi_x)_x \cong \left[\left\{-k\omega C_g + i\omega(C_g)_x\right\} A^* + 2i\omega C_g A_x^*\right] e^{i\psi} \tag{2.11}$$

Finally, using Eq. (2.4),

$$(CC_g \phi_y)_y = (CC_g A_y^*)_y e^{i\psi} \tag{2.12}$$

Substituting Eq. (2.11) and (2.12) into Eq. (2.7) gives:

$$A_x^* - \frac{i}{2\omega C_g}(CC_g A_y^*)_y + A^* \frac{(C_g)_x}{2C_g} + A^* \frac{W}{2C_g} = 0 \tag{2.13}$$

Now, suppose:

$$\phi = A(x,y)\, e^{ik_o x} \tag{2.14}$$

where k_o is a reference wave number and A (x,y) is a slowly varying complex variable. It follows that:

$$\begin{aligned} A^*(x,y) &= A(x,y)\, e^{i\theta} \\ \theta &= k_o x - \int^{x} k dx \end{aligned} \tag{2.15}$$

Thus, Eq. (2.13) can be rewritten as:

$$A_x - i(k - k_o)A + \frac{A}{2C_g}(C_g)_x$$

$$- \frac{i}{2\omega C_g}(CC_g A_y)_y + \frac{W}{2C_g}A = 0 \qquad (2.16)$$

Eq. (2.16) is the simplest parabolic approximation to the elliptic mild-slope equation. It is valid for waves propagating along a predominant direction (+x-axis) or within a small angle to the x-axis. The reference wave number k_o is used as the average wave number along the y-axis.

Kirby (1986) extended Eq. (2.16) to the case of waves propagating at a large angle to the assumed wave direction (x-axis). He derived the following equation:

$$A_x + i(k_o - \beta_1 k)A + \frac{A}{2C_g}(C_g)_x + \frac{\sigma_1}{\omega C_g}(CC_g A_y)_y$$

$$+ \frac{\sigma_2}{\omega C_g}(CC_g A_y)_{yx} + \frac{W}{2C_g}A = 0 \qquad (2.17)$$

where

$$\sigma_1 = i\left(\beta_2 - \beta_3 \frac{k_o}{k}\right) + \beta_3\left(\frac{k_x}{k^2} + \frac{(C_g)_x}{2kC_g}\right)$$

$$\sigma_2 = -\beta_3 / k \qquad (2.18)$$

The coefficients β_1 , β_2 and β_3 are given for different parabolic approximations in Table 2.1. A discussion on the derivation of these coefficients is presented in Section 3.

Table 2.1 Coefficients of the rational approximation determined by varying aperture width

Aperture	B_1	β_2	β_3
Simple	1	- 0.5	0
Padé	1	- 0.75	- 0.25
10^o	0.999999972	- 0.752858477	- 0.252874920
20^o	0.999998178	- 0.761464683	- 0.261734267
30^o	0.999978391	- 0.775898646	- 0.277321130
40^o	0.999871128	- 0.796244743	- 0.301017258
50^o	0.999465861	- 0.822482968	- 0.335107575
60^o	0.998213736	- 0.854229482	- 0.383283081
70^o	0.994733030	- 0.890064831	- 0.451640568
80^o	0.985273164	- 0.925464479	- 0.550974375
90^o	0.956311082	- 0.943396628	- 0.704401903

Eq. (2.17) is the basic equation solved in the parabolic mild-slope module, MIKE 21 PMS. It may be called the parabolic mild-slope equation, since it is a parabolic approximation to the mild-slope equation.

3 Coefficients in Parabolic Mild-Slope Equation

In this section, the coefficients in the various parabolic approximations shown in Table 1 are formally linked to Padé approximants and minimax approximations. This is done by considering the case of linear waves propagating in an area of constant water depth. For this situation, Eq. (2.1) reduces to the Helmholtz equation, assuming no dissipation:

$$\nabla^2\phi + k^2\phi = 0 \tag{3.1}$$

Now, assuming linear waves travelling in a predominant direction (x-axis), the surface wave potential is:

$$\phi = A(x,y)\,e^{ikx} \tag{3.2}$$

Now, substituting Eq. (3.2) in Eq. (3.1), and neglecting second order derivative terms in x, a simple parabolic approximation to Eq. (3.1) is obtained:

$$2ikA_x + A_{yy} = 0 \tag{3.3}$$

Following the procedure of Kirby (1986), the parabolic approximation above, Eq. (3.3), is examined in light of the plane wave of permanent form:

$$\eta = ae^{i(lx+my)}\cdot e^{-i\omega t} \tag{3.4}$$

where

$$l^2 + m^2 = k^2 \tag{3.5}$$

Using Eq. (2.3) and Eq. (3.4):

$$\phi = ae^{i(lx+my-\pi/2)} \tag{3.6}$$

Thus,

$$A(x,y) = ae^{i[(1-k)x+my-\pi/2]} \tag{3.7}$$

Substituting in Eq. (3.3) gives:

$$\frac{l}{k} = 1 - \frac{1}{2}\left(\frac{m}{k}\right)^2 \tag{3.8}$$

Eq. (3.8) is the lowest order binomial expansion (or the (1,0) Padé approximant) of:

$$\frac{l}{k} = \left\{1 - \left(\frac{m}{k}\right)^2\right\}^{1/2} \tag{3.9}$$

This approximation, Eq. (3.8), is good for m/k = sin $\theta \ll 1$, θ being the propagation direction. Kirby (1986) showed that the error in this approximation is small when sin θ < 0.4.

One way of extending the accuracy of a polynomial expansion such as Eq. (3.9) is to construct a Padé approximant of the function. The Padé approximation has the property of predicting the proper value and slope of the approximated function l/k as m/k (or θ) becomes small, while at the same time extending the accuracy of the approximating function as θ increases. For Eq. (3.9) the (1,1) Padé approximant is given by:

$$\frac{l}{k} = \frac{1 - \frac{3}{4}\left(\frac{m}{k}\right)^2}{1 - \frac{1}{4}\left(\frac{m}{k}\right)^2} \tag{3.10}$$

or

$$2k(l-k) + m^2 - \frac{1}{2k}(l-k)m^2 = 0 \tag{3.11}$$

Using Eq. (3.7)

$$\begin{aligned} &= i(l-k)A \\ &= imA \\ &= -m^2A \\ &= -i(l-k)m^2A \end{aligned} \tag{3.12}$$

Thus, using the method of operator correspondence, Eq. (3.11) can be written as:

$$2ikA_x + A_{yy} + \frac{i}{2k}A_{xyy} = 0 \tag{3.13}$$

Eq. (3.13) is the (1,1) Padé approximation of Eq. (3.1). Kirby (1986) showed that the errors in the (1,1) Padé approximation are small when sin θ < 0.65, or θ ≤ 40°.

In order to further extend the accuracy of the parabolic approximation as θ increases, Kirby (1986) used minimax approximations. This is written as:

$$\frac{l}{k} = \frac{\beta_1 + \beta_2\left(\frac{m}{k}\right)^2}{1 + \beta_3\left(\frac{m}{k}\right)^2} \tag{3.14}$$

The minimax method of approximation consists of calculating the coefficients (β_1 , β_2 and β_3), which minimizes the maximum error (l/k - cos θ) over a specified aperture ($0 \le \theta \le \theta_a$). Note that minimax approximations minimize the maximum error in the specified aperture width ($0 < \theta < \theta_a$). However, while reducing the error as θ increases, it may give noticeable errors as θ → 0 in some cases. Kirby (1986) showed that the errors as θ → 0 become noticeable when θa > 60°.

Using the method of operator correspondence, the minimax approximation, Eq. (3.14), leads to the following parabolic approximation:

$$2ikA_x + 2k^2(\beta_1 - 1)A + 2(\beta_3 - \beta_2)A_{yy} - \frac{2i\beta_3}{k}A_{xyy} = 0 \tag{3.15}$$

Notice that Eq. (3.14) becomes (3.15) for constant water depth and no dissipation, as would be expected.

4 Wave Dissipation

The dissipation function W in the parabolic mild-slope equation is calculated as:

$$W = W_b + W_f \tag{4.1}$$

where W_b and W_f are the dissipation functions due to wave breaking and bottom friction respectively. In the following sections the expressions for the dissipation functions are presented.

4.1 Wave Breaking

The dissipation function W_b due to wave breaking is calculated using the method of Battjes and Janssen (1978). They expressed the rate of wave energy dissipation as:

$$E_{diss} = \frac{-\alpha}{8\pi} Q_b \cdot \frac{2\pi}{T_m} \cdot H_{\max}^2 \tag{4.2}$$

$$W_b = E_{diss}/E \tag{4.3}$$

where

$$\frac{1-Q_b}{\ln(Q_b)} = -\left(\frac{H_{rms}}{H_{\max}}\right)^2 \tag{4.4}$$

$$H_{\max} = \gamma_1 k^{-1} \tanh\left(\gamma_2 kd/\gamma_1\right) \tag{4.5}$$

$$E = H_{rms}^2/8 \tag{4.6}$$

In the equation above, α controls the rate of energy dissipation, Q_b is the percentage of breaking waves in the irregular (Rayleigh distributed) wave train, T_m is the energy-averaged mean wave period. H_{max} is the maximum wave height before breaking, H_{rms} is the root mean square wave height, k is the wave number, d is the water depth, γ_1 is a factor controlling the maximum wave steepness allowed before breaking and γ_2 is a factor controlling the maximum H/d allowed before breaking.

The expressions above are used for random waves. For monochromatic waves, the fraction of breaking waves, Qb is taken as 0 (non-breaking waves, $H < H_{max}$) or 1 (breaking waves, $H \geq H_{max}$).

4.2 Bottom Friction

The rate of energy dissipation due to bottom friction is formulated using the quadratic friction law to represent bottom shear stress. For monochromatic waves, Putnam and Johnson (1949) derived:

$$E_{diss} = \frac{-1}{6\pi}\frac{c_{fw}}{g}\left(\frac{\omega H}{\sinh kd}\right)^3 \tag{4.7}$$

where C_{fw} is a wave friction coefficient, H is the wave height and ω is the circular wave frequency.

Dingemanns (1983) extended Eq. (4.7) to the case of unidirectional random waves (Rayleigh distributed) and obtained:

$$E_{diss} = \frac{-1}{8\sqrt{\pi}}\frac{c_{fw}}{g}\cdot\left(\frac{\omega H_{rms}}{\sinh kd}\right)^3 \tag{4.8}$$

W_f is calculated as:

$$W_f = E_{diss}/E \tag{4.9}$$

where E_{diss} is from Eq. (4.7) (monochromatic waves) or Eq. (4.8) (random waves) and E is from Eq. (4.6).

5 Random and Directional Waves

In most practical situations, waves are not regular and long crested. In general, the wave energy is a function of frequency and direction. The distribution of energy with frequency and direction is written as:

$$E(f,\theta) = S(f) \cdot D(f,\theta) \tag{5.1}$$

where

S(f)	an expression for the distribution of energy density with frequency, usually called the frequency spectrum. An example is the JONSWAP spectrum.
D(f,θ)	an expression for the distribution of energy with direction, commonly called the directional spreading function. An example is the $\cos^n(\theta-\theta_0)$ spectrum. Some formulations of the directional spreading function also include a dependence on frequency.
f	frequency
θ	wave direction
$\bar{\theta}$	mean wave direction
n	directional spreading index

The total energy, E_1, in the spectrum is found as:

$$E_1 = \iint E(f,\theta)\, df\ d\theta \tag{5.2}$$

For a given significant wave height H_s, peak wave period T_p and mean wave direction, it is possible to use the MIKE 21 Tool for Generating Irregular and Directional Waves (MIKE 21 Toolbox → Waves → Generate Energy Wave Spectrum) to obtain the distribution of energy with frequency and direction E(f,θ) distributed over discrete frequency and direction bands. This distribution would be specified at the offshore boundary (x=0) of the model.

Thus, at the model boundary, a given amount of wave energy is associated with each discrete frequency and discrete direction. Hence, in discrete form, Eq. (5.2) is written as:

$$E_i = \sum_{j=1}^{NFREQ} \sum_{k=1}^{NDIR} E_{j,k}(f,\theta) \Delta f_j \, \Delta \theta_k \tag{5.3}$$

where

NFREQ	number of discrete frequencies
NDIR	number of discrete directions

In the numerical calculation of the wave climate over the study area, each of the discrete energy component represented in Eq. (5.3) is transformed independently using the parabolic mild-slope equation and the results re-assembled at the inshore grid points using the principle of linear superposition. This procedure is described in the following sections.

5.1 Principle of Linear Superposition

Decomposing the mean free surface potential, ϕ as the sum of several components, we can write:

$$\phi = \sum_{n=1}^{NWAVES} \phi_n \tag{5.4}$$

Now, substituting Eq. (5.4) into Eq. (2.1) and neglecting wave dissipation results in:

$$\nabla \cdot \left(CC_g \nabla \left[\Sigma \phi_n \right] \right) + k^2 CC_g \left[\Sigma \phi_n \right] = 0 \tag{5.5}$$

or

$$\Sigma \left\{ \nabla \cdot (CC_g \nabla \phi_n) + k^2 CC_g \phi_n \right\} = 0 \tag{5.6}$$

Thus, by solving Eq. (5.6). (or its parabolic approximation) for each individual wave component ϕ_n , it is possible to use the principle of linear superposition to obtain the combined ϕ using Eq. (5.4).

It should be noted that the transformation from Eq. (5.5) to Eq. (5.6) is strictly valid when CC_g and k^2CC_g do not change with the discrete wave components. This is the case when all the wave components have the same frequency. Thus, the principle of linear superposition as applied here is valid for directional waves with one frequency. It can be assumed that this principle will also be valid when all the frequencies lie within a narrow frequency band. CC_g and k^2CC_g are calculated using the frequency for the discrete wave component considered.

At the offshore boundary (x=0), the mean surface potential for each wave component, ϕ_n is expressed as (see also Eq. (2.4)):

$$\phi_n = H_n \exp\left(i\, k_n \sin\theta_n\, y + \chi_{o,n} \right) \tag{5.7}$$

where

$\lvert H_n \rvert$	modules of H_n (H_n is complex) is the characteristic wave height for the discrete wave component, n
kn	wave number for the discrete wave component
θn	wave direction for the discrete wave component
χo,n	initial phase for the discrete wave component. This is set to zero.

Since we actually solve for A (given by Eq. (2.14)) using Eq. (2.17), the boundary condition at x=0 may be written as:

$$A_n = H_n \exp\left(ik_n \sin\theta_n y\right) \tag{5.8}$$

Now, assuming that the wave energy within each frequency band in Eq. (5.3) will stay within this frequency upon wave refraction, ie there is no transfer of energy across frequency bands, each discrete energy cell represented in Eq. (5.3) may be propagated independently and the results superposed, in the absence of dissipation. Hence, for each discrete frequency, f_j and discrete wave direction, θ_j. The characteristic wave height can be obtained using:

$$|H| = \sqrt{8\, E_{j,k}\, \Delta f_j \Delta\theta_k} \tag{5.9}$$

where

Ej,k the two-dimensional spectral energy density function at j,k

Δfj, the frequency window at the discrete frequency fj, with fj representing frequencies in the range

$$f_j - \Delta f_j / 2 \le f_j + \Delta f_j / 2$$

Δθk the wave direction window at the discrete direction θk, with θk representing directions in the range

$$\theta k - \Delta\theta k / 2 < \theta < \theta k + \Delta\theta k / 2$$

Hence the wave spectrum is discretized into NWAVES (= NFREQ x NDIR) components, where NFREQ is the number of discrete frequencies and NDIR is the number of discrete directions.

5.2 Inclusion of Dissipation

In coastal applications, it is necessary to include wave dissipation due to bottom friction and wave breaking in numerical wave computations. In MIKE 21 PMS, the procedure used for the wave computations is to solve the governing equation, Eq. (2.17) in two steps, namely, a propagation step and a decay step. For the propogation step, Eq. (2.17) is solved excluding the decay term (i.e. W = 0). Thereafter the decay term is included by solving:

$$A_x + \frac{W}{2C_g} A = 0 \tag{5.10}$$

The finite difference equation is:

$$A_{j+1} = A_{j_*} + \Delta x \left(\frac{W}{2C_g}\right) \cdot \frac{1}{2}\left(A_{j_*} + A_{j+1}\right) \tag{5.11}$$

where A_{j*} is the value of A at (j+1) after the propagation step only (the value of A at (j+1) after propagation step is transferred to (j) for the decay step. A similar procedure is used for time varying parabolic problems).

Thus,

$$A_{j+1} = \frac{\left(1 + \frac{\Delta x\, W}{4\, C_g}\right)}{\left(1 - \frac{\Delta x\, W}{4\, C_g}\right)} A_{j*} \tag{5.12}$$

The procedure can be summarized as follows:

1. Starting from the offshore boundary x=0, calculate the complex function ϕ_n for each component, assuming no wave dissipation at the next row, x+Δx. More correctly, we calculate A_n given by Eq. (2.14), in which the rapid variation of ϕ with x has been factored out. A_n is calculated using the parabolic mild-slope equation, Eq. (2.17).

2. Using the principle of linear superposition, the contribution of all the wave components to the total energy is summed, and the total energy and mean period can be found at x+Δx.

3. The dissipation term is now included by adjusting A for each wave component, n, using:

$$A_{new} = \frac{\left(1 + \frac{\Delta x\, W}{4\, C_g}\right)}{\left(1 - \frac{\Delta x\, W}{4\, C_g}\right)} A_{old} \tag{5.13}$$

   where the wave dissipation function, W, is calculated on the basis of the total energy, E, and the energy averaged mean frequency, f_m.

6 Numerical Solution

The parabolic mild-slope equation, Eq. (2.17) is solved using the Crank-Nicholson numerical scheme for parabolic differential equations. The resulting tridiagonal system of equations is solved using the double-sweep algorithm.

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